The Partially Wrapped Normal Distribution for SE(2) Estimation

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***Abstract*—We introduce a novel probability distribution on the group of rigid motions SE(2) and we refer to this distribution as the partially wrapped normal distribution. Describing probabilities on SE(2) is of interest in a wide range of applications, for example, robotics, autonomous vehicles, or maritime navigation. We derive some important properties of this novel distribution and propose an estimation scheme for its parameters based on moment matching. Furthermore, we provide a qualitative comparison to a recently published approach based on the Bingham distribution, and show that there are complementary advantages and disadvantages of the two approaches.**

Keywords*—*SE(2), wrapped normal, circular-linear correlation, directional statistics.

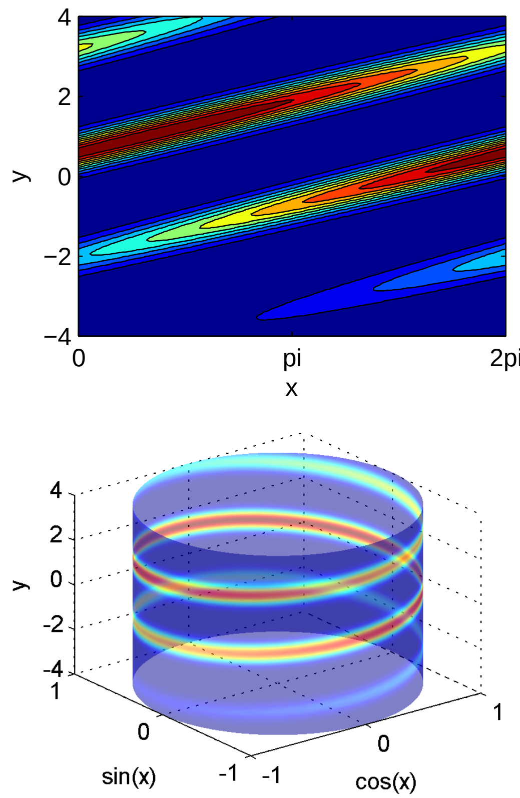

Fig. 1: Probability density of a cylindrical PWN distribution with $n = 2, m = 1$ and parameters $\underline{\mu} = [1, 1]^T, c_{11} = 5^2, c_{12} = c_{21} = 0.99 \cdot 5 \cdot 2.1, c_{22} = 2.1^2$.

I. Introduction

Many applications involve the consideration of the position along with the orientation of an object such as a mobile robot, a car, or a person. In two-dimensional scenarios, position and orientation can be described by the group of rigid body motions in two dimensions $SE(2)$ [1]. However, position and orientation cannot be measured exactly in many practical applications. This problem can be addressed by considering probability distributions on $SE(2)$, which allows describing the uncertainty of an estimate. For this purpose, we propose the use of a novel distribution, which we refer to as the *partially wrapped normal distribution* in this paper.

The new distribution is based on distributions from directional statistics [2], a field in statistics that deals with directional quantities such as angles and orientations. A lot of work has been published on the circular case, for example by Batschelet [3], Fisher [4], and Jammalamadaka [5]. Also, the spherical case has been considered [6], and a variety of spherical distributions have been introduced, e.g., the von Mises-Fisher distribution [7], the Watson distribution [8], the Bingham distribution [9], and the Fisher-Bingham distribution [10].

In order to consider several correlated angles, i.e., taking circular-circular correlation into account, probability distributions on the torus (or more general, the n-torus) have been proposed, particularly the multivariate wrapped normal distribution [11], [12], and the multivariate von Mises distribution [13], [14].

Moreover, the cylindrical case has been considered, i.e., one dimension is periodic whereas the other is not. A circular-linear correlation coefficient has been proposed by Mardia [15], and the wrapped normal distribution [11], [16] as well as the von Mises distribution [17], [18] have been generalized for the cylindrical case.

We have previously investigated a distribution on $SE(2)$ in [19]. There, we proposed a new distribution to be used in conjunction with dual quaternions. This distribution is derived from the Bingham distribution [9] and has a variety of advantages. For example, it is closed under Bayesian inference, but there are also some issues, e.g., a complicated normalization constant. We will present a new distribution for $SE(2)$ that is derived from the wrapped normal distribution. Its strengths and weaknesses are, in a sense, complementary to the distribution proposed in [19] (see Table II).

The contributions of this paper are the following. We define a new probability distribution, which we call partially wrapped normal (PWN) distribution, for arbitrary dimensions and show that it is a generalization of the distributions discussed in [11] and [12]. Particularly, we consider a special case of the PWN distribution that is applicable to $SE(2)$. Furthermore, we define hybrid moments, which can be used to describe random vectors with periodic as well as nonperiodic dimensions. Based on these moments, we derive a parameter estimation scheme. Finally, we discuss the application to $SE(2)$.

II. A New Probability Distribution

In this paper, we build upon two types of basic manifolds, the unit circle S^1 and the real line $\mathbb{R}$. We parameterize the unit circle as $[0, 2\pi) \subset \mathbb{R}$ with the topology of $\{x \in \mathbb{C} : |x| = 1\}$. More general, we consider Cartesian products of these manifolds, i.e., $(S^1)^m \times \mathbb{R}^{n-m}$ for natural numbers $n \geq 1$ and $0 \leq m \leq n$.

A. Prerequisites

For the sake of completeness, we start with a definition of the widely used multivariate normal (or Gaussian) distribution.

Definition 1 (Normal Distribution)**.** An n-dimensional normal distribution is given by the probability density function (pdf)

$$\begin{aligned}&\mathcal{N}(\underline{x}; \underline{\mu}, \mathbf{C})\\&= \frac{1}{(2\pi)^{n/2}|\det \mathbf{C}|} \exp\left(-\frac{1}{2}(\underline{x} - \underline{\mu})^T \mathbf{C}^{-1}(\underline{x} - \underline{\mu})\right)\end{aligned}$$

with $x \in \mathbb{R}^n$, mean $\underline{\mu} \in \mathbb{R}^n$, and symmetric positive definite covariance matrix $\mathbf{C} \in \mathbb{R}^{n \times n}$.

A one-dimensional Gaussian can be wrapped around the circle, which yields the wrapped normal distribution.

Definition 2 (Wrapped Normal Distribution)**.** A wrapped normal (WN) distribution [5] is given by the pdf

$$f(x; \mu, \sigma) = \sum_{k=-\infty}^{\infty} \mathcal{N}(x + 2\pi k; \mu, \sigma^2)$$

with $x \in [0, 2\pi)$, location parameter $\mu \in [0, 2\pi)$, and uncertainty parameter $\sigma > 0$.

B. Partially Wrapped Normal Distribution

Now, we introduce a new distribution, which we refer to as *partially wrapped normal distribution*. It is obtained by taking an n-dimensional Gaussian distribution and wrapping the first $m \leq n$ dimensions.

Definition 3 (Partially Wrapped Normal Distribution)**.** An n-dimensional PWN distribution with $0 \leq m \leq n$ wrapped dimensions is given by the pdf

$$f(\underline{x}; \underline{\mu}, \mathbf{C}) = \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_m=-\infty}^{\infty} \mathcal{N}\left(\underline{x} + \begin{bmatrix} 2\pi k_1 \\ \vdots \\ 2\pi k_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}; \underline{\mu}, \mathbf{C}\right)$$

with $\underline{x} \in [0, 2\pi)^m \times \mathbb{R}^{n-m}$, location parameter $\underline{\mu} \in [0, 2\pi)^m \times \mathbb{R}^{n-m}$, and symmetric positive definite uncertainty parameter $\mathbf{C} \in \mathbb{R}^{n \times n}$.

The PWN distribution is a generalization of both the normal distribution and the WN distribution. In fact, there is a variety of interesting special cases of the PWN distribution that have previously been discussed in literature. We give a taxonomy in Table I. An example of the pdf for the cylindrical case is depicted in Fig. 1.

C. Special Case $SE(2)$

In this paper, we focus on the special case $n = 3, m = 1$, i.e., the group of two-dimensional rigid body motions $SE(2)$. From this point on, we always assume $n = 3, m = 1$ unless otherwise specified. It should be noted that the group of three-dimensional rigid body motions $SE(3)$ does not appear as a special case of the PWN distribution.

In the $SE(2)$ case, the pdf simplifies to

$$f(\underline{x}; \underline{\mu}, \mathbf{C}) = \sum_{k=-\infty}^{\infty} \mathcal{N}\left(\underline{x} + \begin{bmatrix} 2\pi k \\ 0 \\ 0 \end{bmatrix}; \underline{\mu}, \mathbf{C}\right)$$

with $\underline{x} \in S^1 \times \mathbb{R}^2$, $\underline{\mu} \in S^1 \times \mathbb{R}^2$, and symmetric positive definite $\mathbf{C} \in \mathbb{R}^{3 \times 3}$.

The parameters $\underline{\mu}$ and $\mathbf{C}$ do not retain their traditional meaning as mean and covariance. However, they still possess a quite intuitive interpretation. We denote

$$\underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ * & c_{22} & c_{23} \\ * & * & c_{33} \end{bmatrix},$$

and omit symmetric entries in the covariance matrix (marked with asterisks). This leads to the following interpretations:

μ_1	circular mean of periodic part
μ_2, μ_3	mean of linear part
c_{11}	uncertainty of periodic part
c_{12}, c_{13}	circular-linear correlation
$\begin{bmatrix} c_{22} & c_{23} \\ * & c_{33} \end{bmatrix}$	covariance of linear part

D. Marginal Distributions

We now consider the marginal distributions of the PWN distribution and show that they turn out to be Gaussian and WN distributions respectively.

Lemma 1. Marginalization of the circular part yields a normal distribution.

Periodic	Manifold	Distribution	n	m	References
no	real vector space $\mathbb{R}^n$	Gaussian	n	0	[20], [21], [22]
yes	circle S^1	WN	1	1	[2], [23]
yes	torus T^1	bivariate WN	2	2	[11], [12]
yes	n-torus T^n	multivariate WN	n	n	[12]
partial	cylinder $S_1 \times \mathbb{R}$	-	2	1	[11]
partial	SE(2) $S_1 \times \mathbb{R}^2$	-	3	1	this paper

TABLE I: Interesting special cases of the PWN distribution.

Proof: We marginalize circular part according to

$$\begin{aligned} f(x_2, x_3) &= \int_0^{2\pi} f((x_1, x_2, x_3)^T; \underline{\mu}, \mathbf{\Sigma}) dx_1 \\ &= \int_0^{2\pi} \sum_{k=-\infty}^{\infty} \mathcal{N}\left(\begin{bmatrix} x_1 + 2\pi k \\ x_2 \\ x_3 \end{bmatrix}; \underline{\mu}, \mathbf{\Sigma} \right) dx_1 \\ &\overset{(a)}{=} \sum_{k=-\infty}^{\infty} \int_0^{2\pi} \mathcal{N}\left(\begin{bmatrix} x_1 + 2\pi k \\ x_2 \\ x_3 \end{bmatrix}; \underline{\mu}, \mathbf{\Sigma} \right) dx_1 \\ &\overset{(b)}{=} \int_{-\infty}^{\infty} \mathcal{N}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}; \underline{\mu}, \mathbf{\Sigma} \right) dx_1 \\ &\overset{(c)}{=} \mathcal{N}\left(\begin{bmatrix} x_2 \\ x_3 \end{bmatrix}; \begin{bmatrix} \mu_2 \\ \mu_3 \end{bmatrix}, \begin{bmatrix} c_{22} & c_{23} \\ c_{32} & c_{33} \end{bmatrix} \right) . \end{aligned}$$

At (a), we use the dominated convergence theorem, at (b) we use the concatenation of integrals (see [12, Appendix]), and at (c) we use the Gaussian marginal as given in [21, 8.1.2]. □

Lemma 2. Marginalizing the linear part yields a WN distribution.

Proof: We marginalize the linear part according to

$$\begin{aligned} f(x_1) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f((x_1, x_2, x_3)^T; \mu, \Sigma) dx_2 dx_3 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \mathcal{N}\left(\begin{bmatrix} x_1 + 2\pi k \\ x_2 \\ x_3 \end{bmatrix}; \underline{\mu}, \mathbf{\Sigma} \right) dx_2 dx_3 \\ &\overset{(a)}{=} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{N}\left(\begin{bmatrix} x_1 + 2\pi k \\ x_2 \\ x_3 \end{bmatrix}; \underline{\mu}, \mathbf{\Sigma} \right) dx_2 dx_3 \\ &\overset{(b)}{=} \sum_{k=-\infty}^{\infty} \mathcal{N}(x_1 + 2\pi k; \mu_1, c_{11}) . \end{aligned}$$

At (a), we use the dominated convergence theorem, and at (b) we use the Gaussian marginal as given in [21, 8.1.2]. □

E. Conditional Distributions

In this section, we consider what happens when we condition on the linear or the circular part.

Lemma 3. Conditioning a PWN distribution on the linear part yields a WN distribution.

Proof: A direct calculation yields

$$\begin{aligned} f(x_1 | x_2, x_3) &= \frac{f(x_1, x_2, x_3)}{f(x_2, x_3)} \\ &\overset{(a)}{=} \frac{\sum_{k=-\infty}^{\infty} \mathcal{N}\left(\underline{x} + (2\pi k, 0, 0)^T; \underline{\mu}, \mathbf{C} \right)}{\mathcal{N}\left(\begin{bmatrix} x_2 \\ x_3 \end{bmatrix}; \begin{bmatrix} \mu_2 \\ \mu_3 \end{bmatrix}, \begin{bmatrix} c_{22} & c_{23} \\ c_{32} & c_{33} \end{bmatrix} \right)} \\ &\overset{(b)}{=} \sum_{k=-\infty}^{\infty} \frac{\mathcal{N}\left(\underline{x}; \underline{\mu} + (2\pi k, 0, 0)^T, \mathbf{C} \right)}{\mathcal{N}\left(\begin{bmatrix} x_2 \\ x_3 \end{bmatrix}; \begin{bmatrix} \mu_2 \\ \mu_3 \end{bmatrix}, \begin{bmatrix} c_{22} & c_{23} \\ c_{32} & c_{33} \end{bmatrix} \right)} \\ &\overset{(c)}{=} \sum_{k=-\infty}^{\infty} \mathcal{N}(x_1, \tilde{\mu}, \tilde{c}) , \end{aligned}$$

$$\text{with } \tilde{\mu} = \mu_1 + (c_{12}, c_{13}) \begin{bmatrix} c_{22} & c_{23} \\ c_{32} & c_{33} \end{bmatrix}^{-1} \begin{bmatrix} x_2 - \mu_2 \\ x_3 - \mu_3 \end{bmatrix}$$

$$\text{and } \tilde{c} = c_{11} - (c_{12}, c_{13}) \begin{bmatrix} c_{22} & c_{23} \\ c_{32} & c_{33} \end{bmatrix}^{-1} \begin{bmatrix} c_{12} \\ c_{13} \end{bmatrix}$$

where we use Lemma 1 at (a), symmetry of μ and x at (b), and [21, 8.1.3] at (c). □

Lemma 4. Conditioning a PWN distribution on the circular part yields a quotient of infinite series.

Proof: We calculate

$$\begin{aligned} f(x_2, x_3 | x_1) &= \frac{f(x_1, x_2, x_3)}{f(x_1)} \\ &\overset{(a)}{=} \frac{\sum_{k=-\infty}^{\infty} \mathcal{N}\left(\underline{x} + (2\pi k, 0, 0)^T; \underline{\mu}, \mathbf{C} \right)}{\sum_{k=-\infty}^{\infty} \mathcal{N}(x_1 + 2\pi k; \mu_1, c_{11})} , \end{aligned}$$

where we use Lemma 2 at (a). □

III. Moments

In this section, we introduce linear and circular moments, which are then generalized to be applicable to the PWN distribution. We call these generalizations *hybrid moments*.

A. Linear, Circular, and Hybrid Moments

Definition 4 (Linear Moment)**.** For a random variable X defined on $\mathbb{R}$, the l-th linear moment is given by

$$m_l = \mathbb{E}\left(X^l \right) \in \mathbb{R} .$$

Furthermore,

$$m_l^c = \mathbb{E}\left((X - \mathbb{E}(X))^l \right) \in \mathbb{R}$$

is the l-th central linear moment.

If $\underline{X}$ is defined on $\mathbb{R}^n$, the first linear moment (i.e., the mean) is given by $\underline{\mu} = \mathbb{E}(\underline{X})$ and the second central moment (i.e., the covariance) is given by $\mathbf{C} = \mathbb{E}\left((\underline{X} - \underline{\mu}) \cdot (\underline{X} - \underline{\mu})^T\right)$, which simplifies to the second (non-central) moment $\mathbf{C} = \mathbb{E}\left(\underline{X} \cdot \underline{X}^T\right)$ for $\underline{\mu} = \underline{0}$.

The linear correlation is encoded in the off-diagonal entries of $\mathbf{C}$.

Definition 5 (Circular Moment)**.** For a random variable X defined on the circle, the l-th circular moment is given by

$$\begin{aligned} m_l &= \mathbb{E}(\exp(inX)) = \mathbb{E}(\cos(lX) + i\sin(lX)) \\ &= \int_0^{2\pi} f(x)\cos(lx)dx + i\int_0^{2\pi} f(x)\sin(lx)dx \in \mathbb{C} \ . \end{aligned}$$

It should be noted that the first circular moment contains information about both location and uncertainty of the considered distribution. In fact, the circular mean is given by $\arg m_1 = \text{atan2}(\operatorname{Im} m_1, \operatorname{Re} m_1)$ and the uncertainty is quantified by $|m_1| = \sqrt{(\operatorname{Im} m_1)^2 + (\operatorname{Re} m_1)^2}$.

For convenience, we write the l-th circular moment as a two-dimensional real-valued vector

$$[\operatorname{Re} m_l, \operatorname{Im} m_l]^T = [\mathbb{E}(\cos(lX)), \mathbb{E}(\sin(lX))]^T$$

in this paper. Now, it is easily seen that for a circular random variable X, the first circular moment is identical to the first linear moment of the vector $[\cos(x), \sin(x)]^T$. This fact motivates the following definition of the, as we will call it, hybrid moment.

Definition 6 (Hybrid Moment)**.** The first hybrid moment of a partially wrapped random variable $\underline{X}$ on $S_1 \times \mathbb{R}^2$ is given by

$$\underline{m}_1 = \mathbb{E}\left(\begin{bmatrix}\cos(X_1)\\ \sin(X_1)\\ X_2\\ X_3\end{bmatrix}\right) \in \mathbb{R}^4 \ .$$

The second hybrid moment $\mathbf{m}_2 \in \mathbb{R}^{4\times 4}$ of a partially wrapped random variable $\underline{X}$ on $S_1 \times \mathbb{R}^2$ is given by

$$\mathbf{m}_2 = \mathbb{E}\left(\left(\begin{bmatrix}\cos(X_1)\\ \sin(X_1)\\ X_2\\ X_3\end{bmatrix} - \underline{m}_1\right)\left(\begin{bmatrix}\cos(X_1)\\ \sin(X_1)\\ X_2\\ X_3\end{bmatrix} - \underline{m}_1\right)^T\right) \ .$$

This definition is an adaption of the moments considered by Johnson et al. in [11, Sec. 3] for the cylindrical case. It can easily be generalized for PWN distributions arbitrary dimension.

B. Hybrid Moments of the PWN Distribution

It turns out that the first two moments of a PWN distribution can be calculated in closed form.

Lemma 5. The first hybrid moment of a PWN distribution with parameter $\underline{\mu}$ and $\mathbf{C}$ is given by

$$\underline{m}_1 = \underline{\tilde{\mu}} = \begin{bmatrix}\cos(\mu_1)\exp(-c_{11}/2)\\ \sin(\mu_1)\exp(-c_{11}/2)\\ \mu_2\\ \mu_3\end{bmatrix} \ .$$

Proof: A direct calculation yields

$$\begin{aligned} \underline{m}_1 &= \int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{bmatrix}\cos(x_1)\\ \sin(x_1)\\ x_2\\ x_3\end{bmatrix} f(\underline{x}; \underline{\mu}, \mathbf{C}) dx_3 dx_2 dx_1 \\ &= \begin{bmatrix} \int_0^{2\pi}\cos(x_1)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\underline{x}; \underline{\mu}, \mathbf{C}) dx_3 dx_2 dx_1 \\ \int_0^{2\pi}\sin(x_1)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\underline{x}; \underline{\mu}, \mathbf{C}) dx_3 dx_2 dx_1 \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2 \int_0^{2\pi} f(\underline{x}; \underline{\mu}, \mathbf{C}) dx_1 dx_3 dx_2 \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_3 \int_0^{2\pi} f(\underline{x}; \underline{\mu}, \mathbf{C}) dx_1 dx_3 dx_2 \end{bmatrix} \\ &\overset{(a)}{=} \begin{bmatrix} \int_0^{2\pi}\cos(x_1) f(x_1; \mu_1, c_{11}) dx_1 \\ \int_0^{2\pi}\sin(x_1) f(x_1; \mu_1, c_{11}) dx_1 \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2 f(x_{2:3}; \mu_{2:3}, \mathbf{C}_{2:3\times 2:3}) dx_3 dx_2 \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_3 f(x_{2:3}; \mu_{2:3}, \mu_3)^T, \mathbf{C}_{2:3\times 2:3}) dx_3 dx_2 \end{bmatrix} \\ &\overset{(b)}{=} \begin{bmatrix}\cos(\mu_1)\exp(-c_{11}/2)\\ \sin(\mu_1)\exp(-c_{11}/2)\\ \mu_2\\ \mu_3\end{bmatrix} \ , \end{aligned}$$

and use Lemma 1 and Lemma 2 at (a) and the formulas for WN and Gaussian moments at (b).[1] □

Lemma 6. The second hybrid moment of a PWN distribution with parameter $\underline{\mu}$ and $\mathbf{C}$ is given by

$$\mathbf{m}_2 = \tilde{\mathbf{C}} = \begin{bmatrix}\tilde{c}_{11} & \tilde{c}_{12} & \tilde{c}_{13} & \tilde{c}_{14}\\ * & \tilde{c}_{22} & \tilde{c}_{23} & \tilde{c}_{24}\\ * & * & \tilde{c}_{33} & \tilde{c}_{34}\\ * & * & * & \tilde{c}_{44}\end{bmatrix} \ ,$$

where

$$\begin{aligned} \tilde{c}_{11} &= \frac{a}{2}(1 - \exp(-c_{11})\cos(2\mu_1)) \ , \\ \tilde{c}_{22} &= \frac{a}{2}(1 + \exp(-c_{11})\cos(2\mu_1)) \ , \\ \tilde{c}_{12} &= -\frac{a}{2}\exp(-c_{11})\sin(2\mu_1) \ , \\ \tilde{c}_{13} &= -\exp(-c_{11}/2)c_{12}\sin(\mu_1) \ , \\ \tilde{c}_{23} &= \exp(-c_{11}/2)c_{12}\cos(\mu_1) \ , \\ \tilde{c}_{14} &= -\exp(-c_{11}/2)c_{13}\sin(\mu_1) \ , \\ \tilde{c}_{24} &= \exp(-c_{11}/2)c_{13}\cos(\mu_1) \ , \\ \tilde{c}_{33} &= c_{33} \ , \\ \tilde{c}_{34} &= c_{34} \ , \\ \tilde{c}_{44} &= c_{44} \ , \end{aligned}$$

and $a = 1 - \exp(-c_{11})$.

Proof: This is a generalization of the results by Johnson [11, p. 224]. □

For $\mu_1 = 0$, this moment simplifies significantly to

$$\mathbf{m}_2 = \tilde{\mathbf{C}} = \begin{bmatrix}\tilde{c}_{11} & 0 & 0 & 0\\ * & \tilde{c}_{22} & \tilde{c}_{23} & \tilde{c}_{24}\\ * & * & c_{22} & c_{23}\\ * & * & * & c_{33}\end{bmatrix} \qquad (1)$$

[1] The notation $x_{i:j}$ refers to all entries $x_i, x_{i+1}, \ldots, x_{j-1}, x_j$.

where

$$\begin{aligned}
\tilde{c}_{11} &= \frac{1}{2}(1-\exp(-c_{11}))^2 \ , \\
\tilde{c}_{22} &= \frac{1}{2}(1+\exp(-2c_{11})) \ , \\
\tilde{c}_{23} &= \exp(-c_{11}/2)c_{12} \ , \\
\tilde{c}_{24} &= \exp(-c_{11}/2)c_{13} \ .
\end{aligned}$$

C. Sample Moments

In many cases, it is important to consider samples of a probability distribution, for example to estimate the parameters of the distribution from real-world data or to use the distribution in conjunction with particle filtering schemes [24]. For l weighted samples $\underline{s}_1, \dots, \underline{s}_l$ with $\underline{s}_i \in [0, 2\pi) \times \mathbb{R}^2$ and weights $w_1, \dots, w_l$ with $\sum_{k=1}^{l} w_k = 1$, we can calculate the hybrid sample moments as follows. We consider $\underline{\tilde{s}}_i := [\cos(s_{i,1}), \sin(s_{i,1}), s_{i,3}, s_{i,4}]^T$. For the first hybrid moment, we calculate

$$\underline{\tilde{\mu}} = \underline{m}_1 = \sum_{k=1}^{l} w_k \underline{\tilde{s}}_k \ .$$

Furthermore, we obtain the second hybrid moment

$$\tilde{\mathbf{C}} = \mathbf{m}_2 = \sum_{k=1}^{l} w_k (\underline{\tilde{s}}_k - \tilde{\mu})(\underline{\tilde{s}}_k - \tilde{\mu})^T \ .$$

D. PWN Parameter Estimation

Maximum likelihood estimation of the parameters is difficult even for a WN distribution because the likelihood function involves infinite series. For this reason, we rely on fitting a WN to given hybrid moments. More precisely, we fit the first hybrid moment $\tilde{\mu}$, but only certain entries of $\tilde{\mathbf{C}}$, namely $\tilde{c}_{i,j}$ for $i \geq 3, j \geq 3$. This is due to the fact that for the periodic part, $\tilde{\mu}_{1:2}$ encodes the mean and the uncertainty, whereas $\tilde{\mathbf{C}}_{1:2,1:2}$ encodes the second circular moment, which cannot be maintained in general.[2]

For hybrid moments $\tilde{\mu}$ and $\tilde{\mathbf{C}}$, we obtain the PWN parameters $\underline{\mu}$ and $\mathbf{C}$ as follows. According to Lemma 5, we calculate

$$\underline{\mu} = [\operatorname{atan2}(\tilde{\mu}_2, \tilde{\mu}_1), \tilde{\mu}_3, \tilde{\mu}_4]^T$$

and

$$c_{11} = -2\log\left(\sqrt{\tilde{\mu}_1^2 + \tilde{\mu}_2^2}\right) \ .$$

The formula for the first component is identical to formula used for WN parameter estimation in [23]. In order to obtain the entries of $\mathbf{C}$ that encode the linear-circular correlation, we use Lemma 6.

In order to calculate c_{12}, the two equations

$$\begin{aligned}
-\tilde{c}_{13}\exp(c_{11}/2) &= c_{12}\sin(\mu_1) \ , \\
\tilde{c}_{23}\exp(c_{11}/2) &= c_{12}\cos(\mu_1)
\end{aligned}$$

[2]The second circular moment of a WN distribution is a function of the first circular moment.

are given. In the case of a PWN distribution, there is obviously a functional dependence between $\tilde{c}_{13}$ and $\tilde{c}_{23}$. However, this does not hold in general, e.g., for hybrid moments of samples. For this reason, we cannot exactly fulfill both equations and instead seek to minimize the squared error

$$\begin{aligned}
e(c_{12}) &= (p - c_{12}\sin(\mu_1))^2 + (q - c_{12}\cos(\mu_1))^2 \\
&= p^2 + q^2 + c_{12}^2 - 2c_{12}(p\sin(\mu_1) + q\cos(\mu_1))
\end{aligned}$$

with $p = -\tilde{c}_{13}\exp(c_{11}/2)$ and $q = \tilde{c}_{23}\exp(c_{11}/2)$. It holds that

$$\begin{aligned}
\frac{\partial e(c_{12})}{\partial c_{12}} &= 2c_{12} - 2(p\sin(\mu_1) + q\cos(\mu_1)) \\
\frac{\partial^2 e(c_{12})}{(\partial c_{12})^2} &= 2 > 0
\end{aligned}$$

Setting $\frac{\partial e(c_{12})}{\partial c_{12}} = 0$ yields the optimal value

$$c_{12} = p\sin(\mu_1) + q\cos(\mu_1)$$

that minimizes the squared error. Analogously, we calculate

$$c_{13} = \bar{p}\sin(\mu_1) + \bar{q}\cos(\mu_1)$$

where $\bar{p} = -\tilde{c}_{14}\exp(c_{11}/2)$ and $\bar{q} = \tilde{c}_{24}\exp(c_{11}/2)$.

Finally, we copy the linear part

$$\mathbf{C}_{2:3,2:3} = \tilde{\mathbf{C}}_{3:4,3:4} \ .$$

All in all, the proposed procedure exactly retains the entries of the hybrid moment marked in green and approximately retains the entries marked in yellow:

$$\underline{\tilde{\mu}} = \begin{bmatrix} \tilde{\mu}_1 \\ \tilde{\mu}_2 \\ \tilde{\mu}_3 \\ \tilde{\mu}_4 \end{bmatrix} \ , \quad \tilde{\mathbf{C}} = \begin{bmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \tilde{c}_{13} & \tilde{c}_{14} \\ \tilde{c}_{21} & \tilde{c}_{22} & \tilde{c}_{23} & \tilde{c}_{24} \\ \tilde{c}_{31} & \tilde{c}_{32} & \tilde{c}_{33} & \tilde{c}_{34} \\ \tilde{c}_{41} & \tilde{c}_{42} & \tilde{c}_{43} & \tilde{c}_{44} \end{bmatrix} \ .$$

E. Circular-Linear Correlation Coefficient

In 1976, Mardia et al. proposed a circular-linear correlation coefficient [15], which has also been used by [3] and [11]. The circular-linear correlation coefficient of a real variable x and a circular variable θ is obtained by calculating the squared multiple correlation of x and $[\cos(\theta), \sin(\theta)]^T$. For $\mathbb{E}(x) = 0, \mathbb{E}(\sin(\theta)) = 0$ (i.e., linear and circular mean are 0), this yields

$$R^2 = \frac{r_{xc}^2 + r_{xs}^2 - 2r_{xc}r_{xs}r_{cs}}{1 - r_{cs}^2} \in [0, 1] \ , \tag{2}$$

where $r_{xc} = \rho(x, \cos(x))$, $r_{xs} = \rho(x, \sin(x))$, and $r_{cs} = \rho(\cos(x), \sin(x))$ are Pearson correlation coefficients defined according to

$$\rho(x, y) = \frac{\mathbb{E}(x \cdot y)}{\sqrt{\operatorname{Var}(x)\operatorname{Var}(y)}} \ .$$

An obvious disadvantage of this definition is the fact that the sign of the correlation coefficient is lost.

Lemma 7. For a PWN distribution with $\underline{\mu} = 0$, we consider the circular-linear correlation between x_1 and x_2. In this case, we have

$$r_{cs} = \rho(\cos(x_1), \sin(x_1)) \propto \mathbb{E}\left(\cos(x_1) \cdot \sin(x_1)\right) = 0 \ ,$$
$$r_{xc} = \rho(x_2, \cos(x_1)) \propto \mathbb{E}\left(x_2 \cdot \cos(x_1)\right) = 0 \ ,$$

and the circular linear correlation coefficient (2) simplifies to

$$R^2 = r_{xs}^2 \ .$$

Proof: We use the simplification for $\underline{\mu} = 0$ given in (1), which yields

$$\begin{aligned}
0 &= \tilde{c}_{12} \\
&= \mathbb{E}\left((\cos(x_1) - \mathbb{E}\left(\cos(x_1)\right)) \cdot (\sin(x_1) - \mathbb{E}\left(\sin(x_1)\right))\right) \\
&= \mathbb{E}\left((\cos(x_1) - \mathbb{E}\left(\cos(x_1)\right)) \cdot \sin(x_1)\right) \\
&= \mathbb{E}\left(\cos(x_1)\sin(x_1) - \mathbb{E}\left(\cos(x_1)\right)\sin(x_1)\right) \\
&= \mathbb{E}\left(\cos(x_1)\sin(x_1)\right) - \mathbb{E}\left(\cos(x_1)\right)\mathbb{E}\left(\sin(x_1)\right) \\
&= \mathbb{E}\left(\cos(x_1)\sin(x_1)\right) \ ,
\end{aligned}$$

and similarly

$$\begin{aligned}
0 = \tilde{c}_{13} &= \mathbb{E}\left((\cos(x_1) - \mathbb{E}\left(\cos(x_1)\right)) \cdot (x_2 - \mathbb{E}\left(x_2\right))\right) \\
&= \mathbb{E}\left(\cos(x_1) \cdot x_2\right) \ .
\end{aligned}$$

□

More specifically, Lemma 7 yields $|R| = |r_{xs}|$. If we avoid taking the absolute value, we can restore the sign of the correlation in this special case, i.e., $R = r_{xs}$. It can be obtained from the second hybrid moment according to

$$r_{xs} = \frac{\mathbb{E}\left(\sin(x_1) \cdot x_2\right)}{\sqrt{\mathrm{Var}(\sin(x_1))\,\mathrm{Var}(x_2)}} = \frac{\tilde{c}_{23}}{\sqrt{\tilde{c}_{22} \cdot \tilde{c}_{33}}} \ ,$$

where we use the fact that

$$\begin{aligned}
\tilde{c}_{23} &= \mathbb{E}\left((\sin(x_1) - \mathbb{E}\left(\sin(x_1)\right)) \cdot (x_2 - \mathbb{E}\left(x_2\right))\right) \\
&= \mathbb{E}\left(\sin(x_1) \cdot x_2\right) \ .
\end{aligned}$$

We can derive the circular-linear correlation between x_1 and x_3 analogously. For PWN distributions of higher dimension, this approach can be generalized to calculate the correlation between an arbitrary periodic and an arbitrary nonperiodic dimensions. Correlations between two nonperiodic dimensions can be determined just in a classical Gaussian scenario. Correlations between two periodic dimensions are more involved and have been discussed in, e.g., [12], [25].

IV. Application to SE(2) Estimation

In this section, we discuss how the PWN distribution can be applied to SE(2) estimation problems.

A. Parameterization

When applying the PWN distribution to the group of rigid motions SE(2), two different interpretations are possible.

1) Rotate according to x_1, then translate according to $(x_2, x_3)^T$.
2) Translate according to $(x_2, x_3)^T$, then rotate according to x_1.

It is easily seen that both concepts can ultimately be used to represent any position and rotation in $SE(2)$ and, for a fixed pose, it is easy to convert between both interpretations. However, when considering these interpretations in conjunction with a probability distribution on $[0, 2\pi) \times \mathbb{R}^2$ such as the PWN distribution, it makes a significant difference regarding the types of probability distributions that can be expressed. The reason for this effect lies in the fact that two rigid body motions are considered similar when their parameters similar, but the choice of parameterization affects how the parameters relate to rigid body motions, and, thus, which rigid body motions are similar. If rotation is performed first, a small difference in the rotation parameter x_1 can lead to a large difference in the resulting position. If translation is performed first, the resulting position is independent of x_1. In either case, the rotation only depends on x_1. Several examples of the PWN distribution for both interpretations are depicted in Fig. 2 to illustrate this issue.

In homogeneous coordinates [1], rotations and translations are parameterized by

$$R_{x_1} = \begin{bmatrix} \cos(x_1) & -\sin(x_1) & 0 \\ \sin(x_1) & \cos(x_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ T_{x_2,x_3} = \begin{bmatrix} 1 & 0 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix} \ ,$$

and, depending on the order, the combined transformation yields

$$\begin{aligned}
&R_{x_1} \cdot T_{x_2,x_3} \\
&= \begin{bmatrix} \cos(x_1) & -\sin(x_1) & \cos(x_1)x_2 - \sin(x_1)x_3 \\ \sin(x_1) & \cos(x_1) & \sin(x_1)x_2 + \cos(x_1)x_3 \\ 0 & 0 & 1 \end{bmatrix} \ , \\
&T_{x_2,x_3} \cdot R_{x_1} = \begin{bmatrix} \cos(x_1) & -\sin(x_1) & x_2 \\ \sin(x_1) & \cos(x_1) & x_3 \\ 0 & 0 & 1 \end{bmatrix} \ .
\end{aligned}$$

Of course, it is also possible to write these transformations as dual quaternions similar to [19]. However, the PWN distribution is not directly defined on the space of dual quaternions.

B. Operations on the PWN Distribution

In order to derive filtering algorithms, typically two operations are required, addition of random variables (i.e., convolution of densities) and multiplication of densities.

It can be shown that the PWN distribution is closed under convolution and the equations to derive the parameters of the new density are identical to the Gaussian case. More precisely, for two PWN distributions with parameters $\underline{\mu}_1, \mathbf{C}_1$ and $\underline{\mu}_2, \mathbf{C}_2$, the density after convolution is a PWN with parameters $\underline{\mu} = \underline{\mu}_1 + \underline{\mu}_2$ and $\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2$. This property generalizes to arbitrary dimensions.

However, the PWN distribution is not closed under multiplication for $m \geq 1$. This is easy to verify because not even the special case of a WN distribution is closed under multiplication [23]. In order to perform a Bayes update in the context of a stochastic filter, approximations similar to [23] or sampling and reweighting schemes similar to [26] or [27] can be applied.

Based on these results, we give a comparison to the Bingham derivate proposed in [19] in Table II. The PWN inherits the normalization constant and stochastic sampling from the Gaussian distribution, whereas the Bingham derivate's

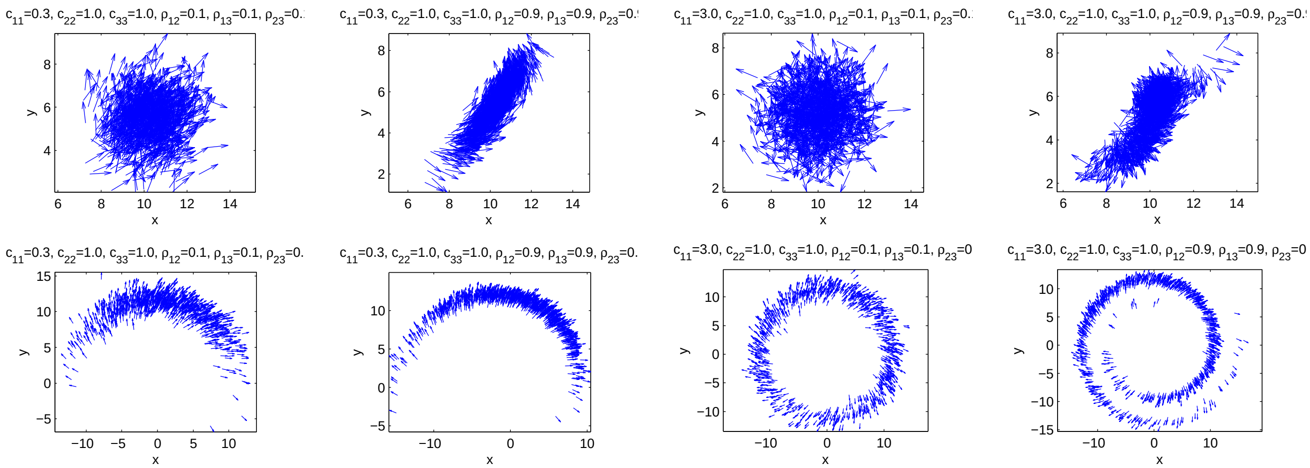

Fig. 2: Samples drawn from exemplary PWN densities on SE(2). We use $\underline{\mu} = [1, 10, 5]^T$ in all cases. In the top row, we apply translation first and rotation later, in the bottom row we do it vice versa. Each arrow indicates the transformation applied to the vector $[1, 0]^T$.

Property	PWN	Bingham derivate [19]
normalization constant	easy	hard
stochastic sampling	easy	hard
moments	easy	hard
antipodally symmetric	no	yes
extensible to $SE(3)$	no	yes
exponential family	no	yes
closed under convolution	yes	no
closed under multiplication	no	yes

TABLE II: Comparison between proposed approach and [19]

normalization constant and sampling are closely related to the significantly more complicated Bingham counterparts. We have derived closed-form equations for the moments of a PWN distribution, but moments of the Bingham derivate are difficult to obtain. However, the Bingham derivate is, just as the Bingham distribution itself, antipodally symmetric and can naturally be applied to dual quaternions allowing both $SE(2)$ and $SE(3)$ applications. Being a member of the exponential family, the Bingham derivate is, unlike the PWN, closed under multiplication. Then again, the PWN is closed under convolution, which is not the case for the Bingham derivate (and neither for the Bingham distribution itself).

V. Conclusion

In this paper, we have presented the partially wrapped normal distribution, a multi-dimensional generalization of the wrapped normal distribution, where some dimensions are wrapped whereas others are not. This distribution has a variety of interesting special cases, particularly it can be applied to the group of two-dimensional rigid body motions $SE(2)$.

We have derived the marginal and conditional distributions. Furthermore, we have introduced the notion of hybrid moments, a generalization of product moments to partially wrapped scenarios. The hybrid moments for PWN distributions have been derived in closed form and a parameter estimation scheme based on moment-matching has been given. Additionally, the circular-linear correlation coefficient [15] was derived analytically for the PWN distribution. Finally, we have discussed the application to SE(2) estimation.

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